THE EULER–LAPLACE THEOREM ON THE DECREASE OF THE ECCENTRICITY OF THE ORBITS OF THE HEAVENLY BODIES UNDER THE SECULAR ACTION OF A RESISTING MEDIUM.

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In the "Mécanique Céleste," Liv. VII., Chap. VI., §§ 29–30, and Liv. X., Chap. VII., § 18, Laplace has developed the mathematical theory of the secular action of a resisting medium, and applied it to the motions of the moon and planets. The first discussion herein cited was published in Volume III. of the "Mécanique Céleste," 1802. It is on this discussion by Laplace that modern investigators chiefly base their treatment of the problems of a resisting medium. Laplace's development of the theory therefore has been of great service to science for more than a century.

Recently, while occupied with a careful review of the theories of magnetism and of gravitation since the time of Newton, I had occasion to examine Euler's "Dissertatio de Magnete," 1744, "Opuscula," 1746–51; and while looking into this work was surprised to find that Euler had preceded Laplace in his development of the chief effects of a resisting medium by more than half a century. Euler's work on the resisting medium will be found in the volume of "Opuscula," Berlin, 1746, in the paper "De Relaxatione Motus Planetarum," pp. 245–276.

Having shown that the aphelia are undisturbed by resistance, Euler considers in section XVII. the equations for the mean motion, and the return to perihelion, after changes in the mean motion by the increments representing a whole revolution:

$$nt + 2\pi, \quad nt + 4\pi, \quad nt + 6\pi, \quad nt + 8\pi, \text{ etc.}$$

Euler puts for the planetary orbit about the sun,

$h = a\,(1 - c) =$ perihelion distance,

$g = a\,(1 - e^2) = p =$ latus rectum of the orbit,

$y = r =$ radius vector of the planet,

$Z = e =$ the eccentricity of the orbit,

$t =$ true anomaly $= v$, in the notation now commonly used,

$s =$ arc of the orbit, reckoned from perihelion,

$c =$ sun's mean distance,

$= \mu\alpha$, where α is the earth's equatorial semi-diameter, and

μ a number which expresses the sun's mean distance in this unit. Euler uses a solar parallax of 13″, and takes $c = 15866\alpha$. With the values now adopted in astronomy we have about $c = 23445\alpha$. In some of his numerical work Euler uses $c = g = a\,(1 - e^2)$, which is admissible when we neglect the square of the eccentricity.

Euler also uses a small angle of deviation due to the angular effects of resistance, $z = \theta$, such that $\tan z = 2g/3c$; and then takes the equation for the Keplerian ellipse

$$r = \frac{a(1 - e^2)}{1 + e\cos v},$$

to have the form of an ellipse modified by resistance

$$\frac{1}{r} = \frac{1}{p}(1 + e\cos v) = \frac{1}{p} + \frac{e\cos v}{p} + P,$$

where P is function of the time, but modified by a very small quantity depending on the effects of the secular action of the resisting medium.

From the equations of the disturbed ellipse, in his notation,

$$\frac{1}{y} = \frac{1}{g} + \frac{\zeta}{g}\cos t + P,$$

$$P = \frac{1}{c}\left(t - \sin t - \tfrac{1}{2}\zeta\sin t + \tfrac{1}{2}\zeta t\cos t + \tfrac{3}{4}\zeta\zeta t - \tfrac{2}{3}\zeta\zeta\sin t - \tfrac{1}{24}\zeta\zeta\sin 2t\right),$$

Euler develops the following table:

If	there will be
$t = 0$,	$P = 0$,
$t = \pi - \theta$,	$P = \frac{1}{c}\left(\pi - \frac{1}{2}\zeta\pi - \frac{4g}{\zeta c}\right)$,
$t = 2\pi$,	$P = \frac{1}{c}(2\pi + \zeta\pi)$,
$t = 3\pi - \theta$,	$P = \frac{1}{c}\left(3\pi - \frac{3}{2}\zeta\pi - \frac{4g}{\zeta c}\right)$,
$t = 4\pi$,	$P = \frac{1}{c}(4\pi + 2\zeta\pi)$,
$t = 5\pi - \theta$,	$P = \frac{1}{c}\left(5\pi - \frac{5}{2}\zeta\pi - \frac{4g}{\zeta c}\right)$.

He remarks that when therefore for perihelion we have

$$\frac{1}{y} = \frac{1 + \zeta}{g} + P,$$

and for aphelion

$$\frac{1}{y} = \frac{1 - \zeta}{g} + P,$$

if	$t = 0$,	$\frac{1}{y} = \frac{1 + \zeta}{g} + 0$,
	$t = \pi - \theta$,	$\frac{1}{y} = \frac{1 - \zeta}{g} + \frac{\pi}{c}(1 - \frac{1}{2}\zeta)$,
	$t = 2\pi$,	$\frac{1}{y} = \frac{1 + \zeta}{g} + \frac{2\pi}{c}(1 + \frac{1}{2}\zeta)$,
	$t = 3\pi - \theta$,	$\frac{1}{y} = \frac{1 - \zeta}{g} + \frac{3\pi}{c}(1 - \frac{1}{2}\zeta)$,
	$t = 4\pi$,	$\frac{1}{y} = \frac{1 + \zeta}{g} + \frac{4\pi}{c}(1 + \frac{1}{2}\zeta)$,
	$t = 5\pi - \theta$,	$\frac{1}{y} = \frac{1 - \zeta}{g} + \frac{5\pi}{c}(1 - \frac{1}{2}\zeta)$;

it being understood that the final angle $4g/3c$ is neglected as very small.

Euler next considers the effect of i whole revolutions:

$$t = 2i\pi,$$

$$\frac{1}{y} = \frac{1+\zeta}{g} + \frac{2i\pi}{c}(1+\tfrac{1}{2}\zeta);$$

and finds for the radius vector:

$$y = \frac{g}{1+\zeta} - \frac{2i\pi}{c}\,\frac{(1+\frac{1}{2}\zeta)gg}{(1+\zeta)^2}.$$

Putting for the following aphelion, $t = (2i+1)\pi - \theta$, there will result

$$\frac{1}{y} = \frac{1-\zeta}{g} + \frac{(2i+1)\pi}{c}(1-\tfrac{1}{2}\zeta);$$

whence the radius vector becomes

$$y = \frac{g}{1-\zeta} - \frac{(2i+1)\pi(1-\frac{1}{2}\zeta)gg}{c(1-\zeta)^2}.$$

The successive distances of the planet from the sun are diminished in the following manner:

I. Perihelion	$\frac{g}{1+\zeta} = 0,$
Aphelion	$\frac{g}{1-\zeta} - \frac{\pi(1-\frac{1}{2}\zeta)gg}{c(1-\zeta)^2},$
II. Perihelion	$\frac{g}{1+\zeta} - \frac{2\pi(1+\frac{1}{2}\zeta)gg}{c(1+\zeta)^2},$
Aphelion	$\frac{g}{1-\zeta} - \frac{3\pi(1-\frac{1}{2}\zeta)gg}{c(1-\zeta)^2},$
III. Perihelion	$\frac{g}{1+\zeta} - \frac{4\pi(1+\frac{1}{2}\zeta)gg}{c(1+\zeta)^2},$
Aphelion	$\frac{g}{1-\zeta} - \frac{5\pi(1-\frac{1}{2}\zeta)gg}{c(1-\zeta)^2}$, etc.

In any revolution about the sun the perihelion advances by the interval

$$\frac{2\pi(1+\frac{1}{2}\zeta)gg}{c(1+\zeta)^2};$$

and the aphelion regresses by the interval

$$\frac{2\pi(1-\frac{1}{2}\zeta)gg}{c(1-\zeta)^2};$$

the mean distance therefore decreases in the interval about $2\pi gg/c$; and after i revolutions this decrease in the mean distance will be $2i\pi gg/c$.

Accordingly, after i planetary revolutions, the perihelion distance from the sun becomes:

$$\frac{g}{1+\zeta}-\frac{2i\pi(1+\frac{1}{2}\zeta)gg}{c(1+\zeta)^2};$$

and the following aphelion distance:

$$\frac{g}{1-\zeta}-\frac{(2i+1)\pi(1-\frac{1}{2}\zeta)gg}{c(1-\zeta)^2}.$$

The addition of these values, after i revolutions, effects the transverse axis of the orbit:

$$\frac{2g}{1-\zeta\zeta}-\frac{4i\pi gg}{c(1-\zeta\zeta)^2}-\frac{\pi(1-\frac{1}{2}\zeta)gg}{c(1-\zeta)^2}.$$

Here indeed, since the time is to be defined, the time from perihelion to aphelion may be omitted; and thus after i revolutions the transverse axis of the orbit is found to be:

$$\frac{2g}{1-\zeta\zeta}-\frac{4i\pi gg}{c(1-\zeta\zeta)^2};$$

wherefore also the initial transverse axis is assumed equal to $2g/(1-\zeta\zeta)$.

If, therefore, the distance from the perihelion to the sun after i revolutions, which is equal to

$$\frac{g}{1+\zeta}-\frac{2i\pi(1+\frac{1}{2}\zeta)gg}{c(1+\zeta)^2},$$

be subtracted from the distance from the aphelion to the sun, which would develop in the same time, and found to be equal to

$$\frac{g}{1-\zeta}-\frac{2i\pi(1-\frac{1}{2}\zeta)gg}{c(1-\zeta)^2},$$

it will give for the distance of the foci after i revolutions

$$\frac{2\zeta g}{1-\zeta\zeta}-\frac{2i\pi\zeta gg(3-\zeta\zeta)}{c(1-\zeta\zeta)^2}.$$

The initial transverse axis was $2g/(1-\zeta\zeta)$, and if we divide this into the last expression, we get for the eccentricity of the orbit at this time: $\zeta - 3i\pi\zeta g/c$, terms in ζ^3 being neglected as insensible.

In Euler's paper the factor 3 in the last term is inadvertently omitted. He remarks that the original eccentricity was ζ, whereas after i revolutions it is decreased by the negative term shown above, and thus is subject to a secular diminution, owing to the secular action of the resisting medium.

After this discussion Euler reaches the conclusion: "A resistentia ergo excentricitas continuo minuitur, orbitaeque planetarum propius ad figuram circularem reducuntur" (p. 271).

He therefore recognized clearly that the effect of a resisting medium is to decrease the eccentricity incessantly, and to render the orbit more and more circular; and had reached this important conclusion some fifty-six years (1746) before the corresponding theorem was established by Laplace in 1802.

Accordingly as Euler's reasoning is essentially rigorous, though not the same as that of Laplace, it is evident that he was the first discoverer of the theorem which is of such fundamental importance in the theories of cosmogony.

It is remarkable that although Laplace had this theorem clearly before his mind for a quarter of a century at the close of his life (1802–1827) he did not once suspect that the planets and satellites had originated in the distance and through the action of a resisting medium had neared the centers about which they now revolve, and thus acquired the wonderful circularity of their orbits.

It is well known that Laplace continually refers to these bodies as detached by rotation, in the form of zones of vapor, as first outlined in his nebular hypothesis of 1796. He thus misled the scientific world for more than a century, till the capture theory, involving formation in the distance with subsequent approach to their central masses, under the secular action of a resisting medium, was developed by the present writer in 1908–10.

It is equally well known that Laplace always held the comets to be foreign to our system—another misleading doctrine in cosmogony, finally overthrown in 1910 by the independent researches of Strömgren of Copenhagen, and the present writer, who showed that the comets are surviving residues of the ancient nebula which formed our solar system.

In my "Researches," Vol. II., pp. 138–139, I have drawn attention to two letters from Euler to the Royal Society, pointing out, as early as 1749, that the earth was once beyond the present orbit of Saturn. He does not there discuss the secular decrease of the eccentricity of the planetary orbits; yet as he had grounds for holding to a secular approach to the central masses, he was the first writer to outline sound views in cosmogony.

Under the circumstances it appears appropriate that the theorem on the secular decrease of the eccentricities of the orbits of bodies moving in resisting media, should be known by the name of the Euler-Laplace theorem. This recognizes the correct historical development, as now made out; and probably will always hold a fundamental place in the science of celestial evolution.

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